# Exploiting Decryption Failures in Mersenne Number Cryptosystems

**Marcel Tiepelt**[1] and Jan-Pieter D'Anvers[2]

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APKC
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# Decryption Failures in Post-Quantum Cryptography

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*Disclaimer:*

- Ramstake

  (Secure?) Round 1 candidate for NIST post-quantum project

# Mersenne Number Cryptosystem

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- Secrets $a, b \in \mathbb{Z}_p$ with *low* Hamming weight
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**Mersenne Low Hamming Combination Problem**

For random $n$-bit string $R$, distinguishing the tuples

$$(G, aG + b \mod p) \text{ or } (G, R)$$

is difficult.

# Mersenne Number Cryptosystem

| **Alice** | Fix Mersenne number $p$, $G \xleftarrow{\$} \mathbb{Z}_p$ | **Bob** |
| --- | --- | --- |
| $a, b \xleftarrow{\$} \mathsf{SMALL}_{\mathsf{HW}}(\mathbb{Z}_p)$ | | $c, d \xleftarrow{\$} \mathsf{SMALL}_{\mathsf{HW}}(\mathbb{Z}_p)$ |
| $P_A \equiv aG + b \mod p$ | | $P_B \equiv cG + d \mod p$ |

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- $(acG + ad)_{[0:|m|]} \approx (acG + bc)_{[0:|m|]}$

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Alice → Bob: $P_A$

Bob:

$$c_m = encode(m)$$
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Bob → Alice: $(ctxt, P_B)$

Alice:

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- $(acG + ad)_{[0:|c_m|]} \approx (acG + bc)_{[0:|c_m|]}$
- $Pr[m \neq m']$ too high $\implies$ introduce ECC
- $encode(\cdot)$, $decode(\cdot)$, correct up to $t$ errors

# Example Parameters

| Ramstake-756839 | |
|---|---|
| Mersenne exponent | $n = 756839$ |
| Hamming weight | 128 |
| #Corrected Errors | $t = 111$ |
| $Pr[m' \neq m]$ | $2^{-64}$ |
| quantum security | 128 |
| $\|pk\|$ | $93kB$ |
| $\|ctxt\|$ | $94kB$ |

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Our Attack: $\approx 2^{46}$ quantum steps + $2^{72}$ decryption queries

# Attacking Ramstake: Slice-and-Dice

Introduced by Beunardeau et al. [Beu+19]

$a$ $G +$ $b$ $= P_A$

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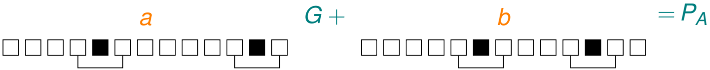

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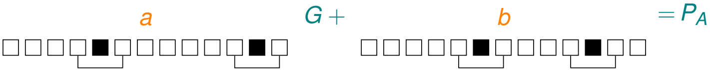

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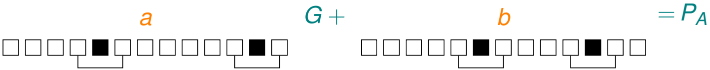


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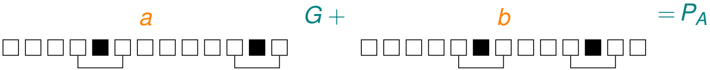


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Decryption failures to make a good guess!

# Ramstake: Decryption Failures

**Alice**

$$c'_m = ctxt \oplus (aP_B \mod p)_{[0:|c_m|]}$$
$$= ctxt \oplus (acG + ad \mod p)_{[0:|c_m|]}$$
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## Decryption Failure

$$decode(c'_m) \text{ fails}$$
$$\Leftrightarrow \mathrm{HW}_{[0:|c_m|]}((acG + ad \mod p) \oplus (acG + bc \mod p)) > t$$
$$\approx \left(\mathrm{HW}_{[0:|c_m|]}(ad) + \mathrm{HW}_{[0:|c_m|]}(bc)\right) > t$$

# Ramstake Information Leak

- Consider only error *ad*
- Assume decryption returns *fail*

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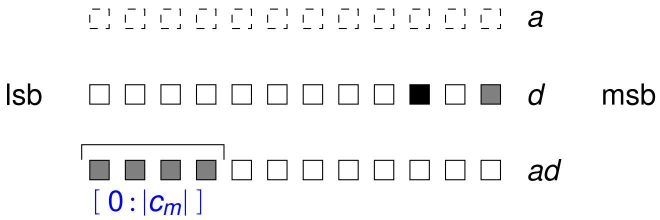

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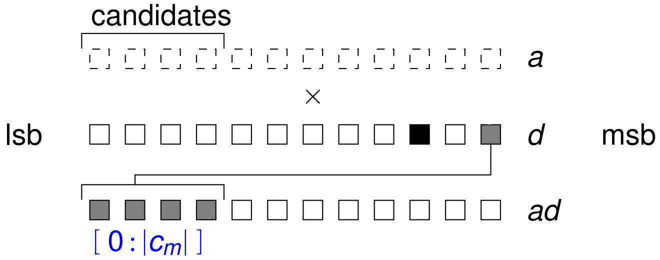

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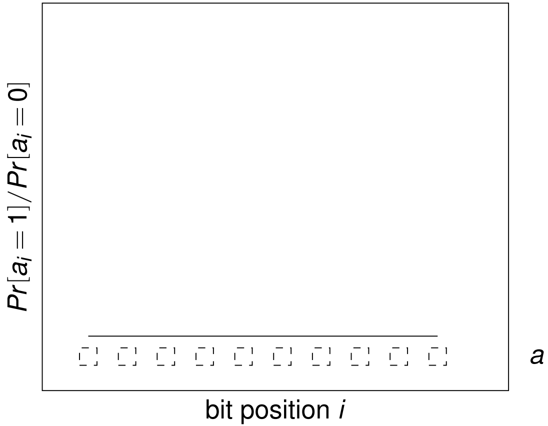


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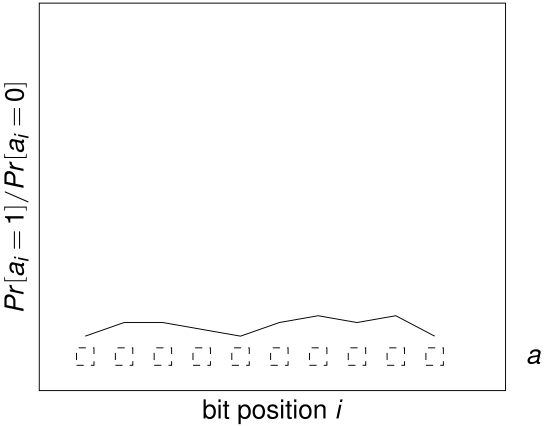


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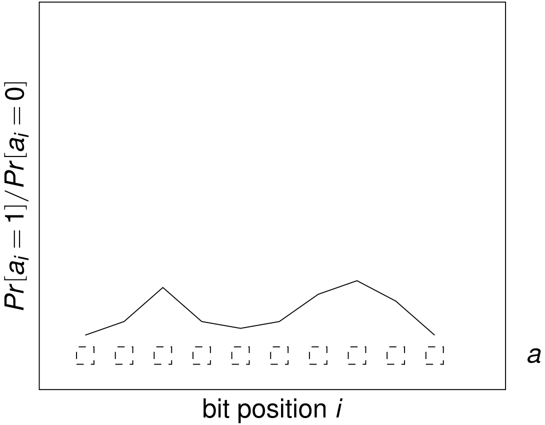


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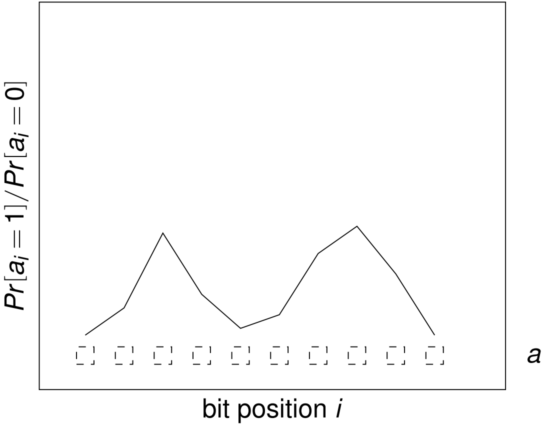


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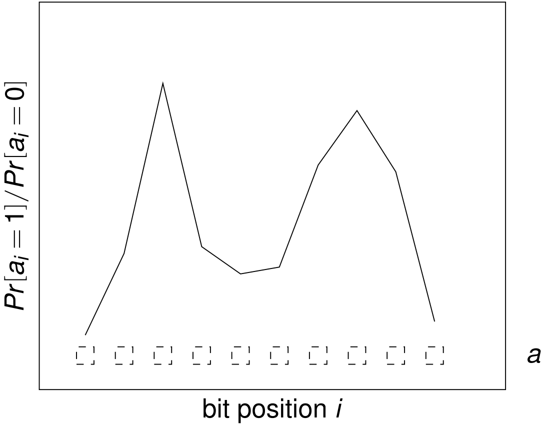


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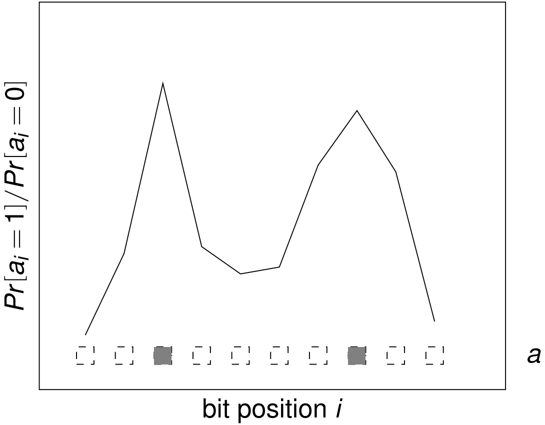


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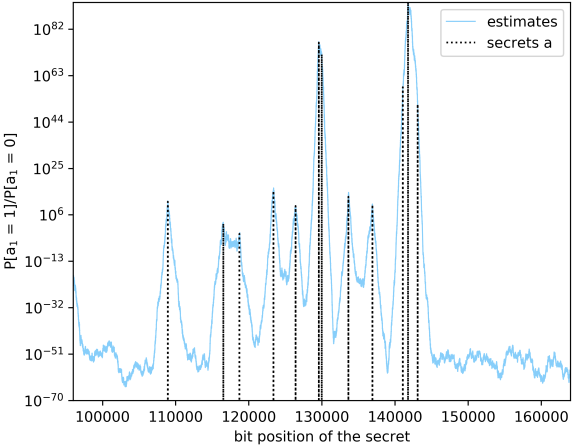
# “Nothing-up-our-sleeves” Result

https://github.com/Fleeep/ramstake-failure-attack



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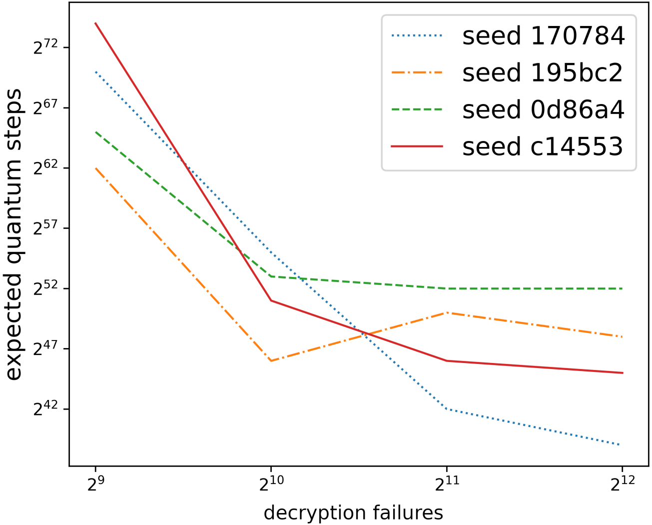
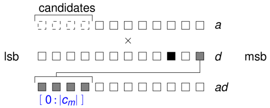



Ramstake-756839
(Security level: 128)

| #decryption failures | approx. # quantum steps |
|---|---|
| $2^9$ | $2^{68}$ |
| $2^{10}$ | $2^{52}$ |
| $2^{11}$ | $2^{48}$ |
| $2^{12}$ | $2^{46}$ |

# Conclusion

Mersenne number cryptosystems
leak information



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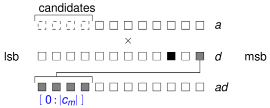


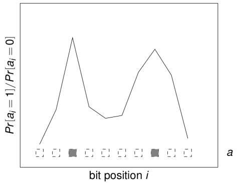

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Mersenne number cryptosystems leak information

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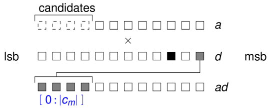
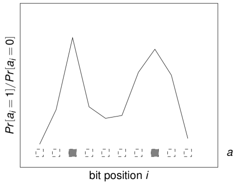
Probability of failure should to be **very low.**



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